Lecture 38

16.6 - Parametric Surfaces and Their Areas.

We've spent time before parametrizing curves, but now it's time to move up a dimension to surfaces. Since curves were only one dimensional, they only took one variable to parametrize; so, since surfaces are 2-dimensional, they'll take 2 variables to parametrize. Given a surface, S, in $\mathbb{R}^3$, a <u>parametrization of S</u> will take the form

$$\vec{r}(u,v) = \langle x(u,v), y(u,v), z(u,v) \rangle \quad , \quad (u,v) \in D$$

where D is the domain of $\vec{r}$, sometimes called the <u>domain of parametrization</u>. One could say that

$$\begin{cases} x = x(u,v) \\ y = y(u,v) \\ z = z(u,v) \end{cases} , \quad (u,v) \in D$$

are the <u>parametric equations of S</u>.

Before trying to figure out our own parametrizations, let's see some examples:

Ex: Identify and sketch the surface parametrized by:

a) $\vec{r}(u,v) = \langle u\cos v, u\sin v, 2\rangle$, $0 \le u \le 1$, $0 \le v \le 2\pi$.

b) $\vec{r}(s,t) = \langle s, \pi, t\rangle$, $s,t \in \mathbb{R}$

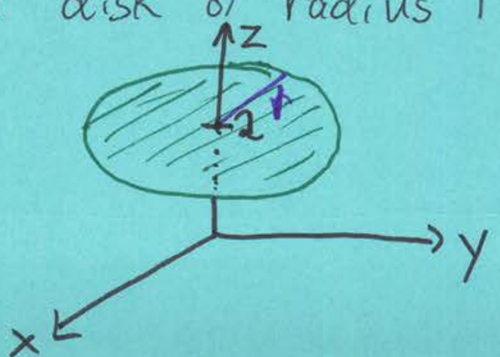
c) $\vec{r}(u,v) = \langle u, v, u^2+v^2\rangle$, $u,v \in \mathbb{R}$

d) $\vec{r}(p,q) = \langle (2+\cos(p))\cos(q), (2+\cos(p))\sin(q), \sin(p)\rangle$
$0 \le p \le 2\pi$, $0 \le q \le 2\pi$

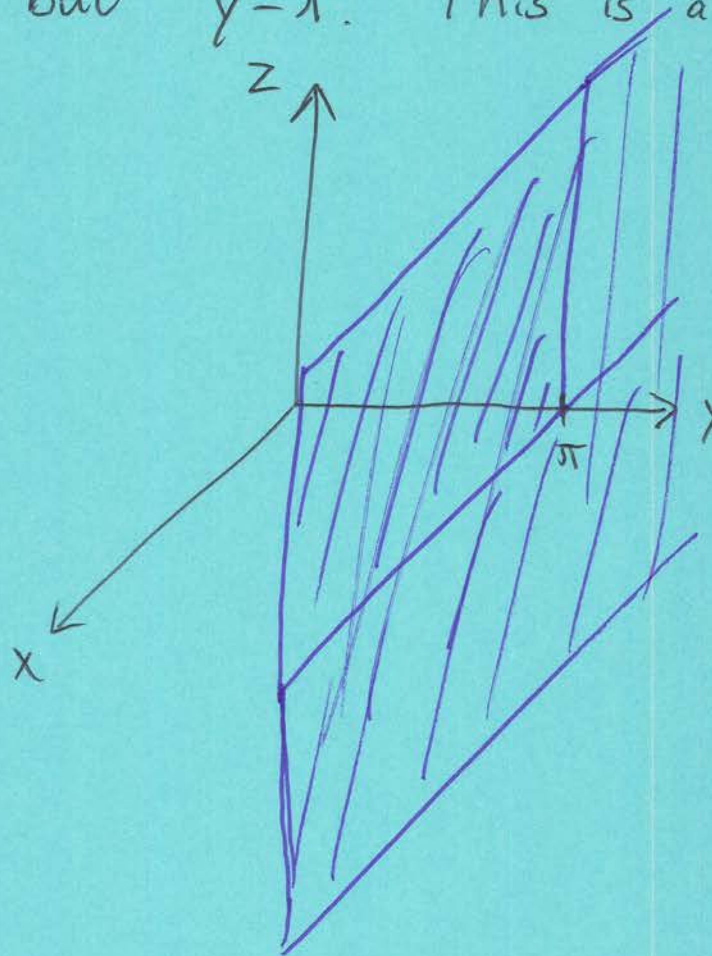
e) $\vec{r}(s,t) = \langle s\cos t, s\sin t, t\rangle$, $0 \le s \le 2$, $0 \le t \le 4\pi$

Sol:

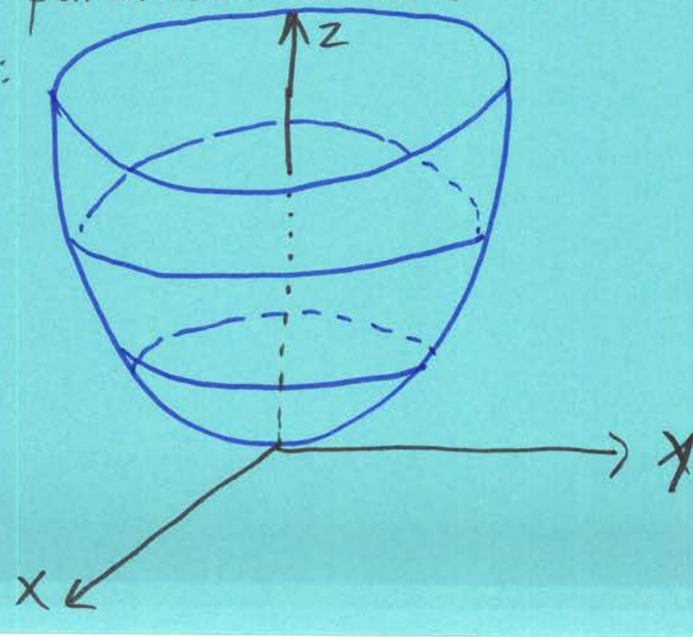
a) Since $x^2+y^2 = (u\cos v)^2 + (u\sin v)^2 = u^2$ is a circle of radius u, and u fills in from 0 to 1, and $z=2$, this is just a disk of radius 1 at height $z=2$:



b) Here s & t can be anything, i.e., x & z can be anything, but $y=\pi$. This is a parametrization of $y=\pi$.



c) We have $x=u$, $y=v$, & $z=u^2+v^2$, i.e., $z=x^2+y^2$. This is a parametrization of the paraboloid $z=x^2+y^2$:



d) This is a torus.

e) This is a helicoid.

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Now that we've seen some examples, let's try parametrizing some on our own. The easiest example is to parametrize a surface which is a graph, i.e.,

$$z = f(x,y).$$

Here we just choose x and y as our parameters and use that to get the z-component, i.e.

$$\vec{r}(x,y) = \langle x, y, f(x,y) \rangle$$

parametrizes the surface.

<u>Ex</u>: Parametrize the surface $z = 3\sqrt{x^2+y^2}$.

<u>Sol</u>: If we choose x and y as our parameters, then we have $\vec{r}(x,y) = \langle x, y, 3\sqrt{x^2+y^2} \rangle$.
In this case, however, it would be easier to use another parametrization, inspired by polar coordinates, since the surface is merely $z = 3r$ in polar.(cylindrical).

Here we have:

$$\vec{r}(r,\theta) = \langle r\cos\theta, r\sin\theta, 3r\rangle, \ r \geq 0, \ 0 \leq \theta \leq 2\pi$$

since $x = r\cos\theta$ & $y = r\sin\theta$.

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Ex: Parametrize the sphere $x^2+y^2+z^2=81$.

Sol: This isn't the graph of a function, however, inspired by the previous example, we have an easy way to parametrize the sphere... using spherical coordinates! In spherical, this sphere is given by $\rho=9$, so set $\rho=9$ in the equations for spherical:

$$\vec{r}(\theta,\varphi) = \langle 9\cos\theta\sin\varphi, 9\sin\theta\sin\varphi, 9\cos\varphi\rangle, \ 0 \leq \theta \leq 2\pi, \ 0 \leq \varphi \leq \pi.$$

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Now, let's revisit a problem from the second week of class: finding the equation of a plane.

We could find the equation of a plane by knowing a point P in the plane and two vectors $\vec{a}$ & $\vec{b}$ in the plane.

We can parametrize said plane by:

$$\vec{r}(s,t) = \overrightarrow{OP} + s\vec{a} + t\vec{b}$$

$\overrightarrow{OP}$: position vector of P

<u>Ex</u>: Parametrize the plane passing through the points $(3,-1,2)$, $(8,2,4)$, and $(-1,-2,-3)$.

<u>Sol</u>: Using our chapter 12 methods, we would find

$$-13x + 17y + 7z = -42$$

For us, if we choose $P = (-1,-2,-3)$, then

$$\overrightarrow{OP} = \langle -1,-2,-3 \rangle, \qquad (Q = (3,-1,2),\ R = (8,2,4))$$

$$\vec{a} = \overrightarrow{PQ} = \langle 4,1,5 \rangle,$$

$$\vec{b} = \overrightarrow{PR} = \langle 9,4,7 \rangle$$

and a parametrization is

$$\vec{r}(s,t) = \overrightarrow{OP} + s\vec{a} + t\vec{b} = \langle -1+4s+9t, -2+s+4t, -3+5s+7t \rangle.$$

If you plug this into the equation above, you'll see it checks.

$\diamond$

Another type of surface we can parametrize easily is a <u>surface of revolution</u>. This is a surface obtained by taking the curve $y=f(x)$ in the xy-plane, and revolving it around the x-axis. This is parametrized by

$$\vec{r}(x,\theta)=\langle x, f(x)\cos\theta, f(x)\sin\theta\rangle.$$

<u>Ex</u>: Parametrize the surface obtained by revolving the curve $y=\frac{1}{x}$, $x\geq 1$, about the x-axis.

<u>Sol</u>: A parametrization of this is:

$$\vec{r}(x,\theta)=\langle x, \tfrac{1}{x}\cos\theta, \tfrac{1}{x}\sin\theta\rangle,\ x\geq 1,\ 0\leq\theta\leq 2\pi.$$

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You may recognize the previous surface as <u>Gabriel's Horn</u>, the surface which bounds a finite volume, but has infinite surface area.